Fractal Approaches in Signal Processing

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Abstract

We review some of the recent advances that have been made in the application of fractal tools for studying complex signals. The first part of the paper is devoted to a brief description of the theoretical methods used. These essentially consists in generalizations of previous well known techniques that allow to efficiently handle real signals. We present some general results dealing with the multifractal analysis of sequences of Choquet capacities, and the possibility of constructing such capacities with prescribed spectrum. Related results concerning the pointwise irregularity of a continuous function at each point are given in the frame of iterated functions systems. Finally, some results on a particular stochastic process are sketched: we define the multifractional Brownian motion, a generalization of the classical fractional Brownian motion, where the parameter H is replaced by a function verifying some regularity conditions. The second part consists in the description of selected applications of current interest, in the fields of image analysis, speech synthesis and road traffic modeling. In each case we try and show how a fractal approach provides new means to solve specific problems in signal processing, sometimes with greater success than classical methods.

1 INTRODUCTION

Applications of the fractal theory in signal processing is fairly recent, compared to the state of the art in other areas of science. Several facts explain this situation:

- Signal processing is a field founded on solid scientific roots, where the methods perform reasonably well in many cases. These methods are generally quite classical and thus far from the fractal schemes.
- A recurrent argument against the use any fractal approach in this field is that "real" signals are not "fractals", and thus that this framework is unadapted.
- Finally, a fact which supports the previous item is that the first (and maybe naive) applications of fractals to signal processing (e.g. image and speech analysis, geophysical time series analysis, etc ...) indeed led to disappointing results, at least in terms of practical improvements.

A few things happened in recent years, that dramatically modify this situation. From a theoretical point of view, more versatile tools have been designed, that allow to describe cases where no strict self similarity exists. As for the practical aspects, new algorithms have been developed that are more robust and give more precise results.

Finally, it has largely been understood that signals need not be "fractal" to be efficiently analyzed by fractal methods. More precisely, in the field of signal processing, as in some other domains, the question is not so much: "Is the signal a fractal?", than: "What are the local scaling properties of the signal, and does there exist a simple, geometrical or statistical, global description of these properties?"

The remainder of this paper tries and answer such questions for a few specific problems in signal processing.

Section 2 presents theoretical tools that have been applied with some success in our field. No attempt is made to be complete, and three distinct topics are delt with. The first one is multifractal analysis: we first recall some basic facts, and then present a few extensions designed to efficiently handle real signals. These include the multifractal analysis of Choquet capacities and the notion of mutual multifractal analysis, which allows to build algorithms yielding precise and robust results.

Moving to a different topic, we develop a generalization of the IFS theory that allows to construct continuous functions whose pointwise Hölder exponent at each point is prescribed.

The last part of this section is devoted to the definition of a new stochastic process, the multifractional Brownian motion, and to the presentation of its main properties.

In section 3, we focus on three specific applications in image analysis, speech synthesis and road traffic modeling. These applications will hopefully show that the fractal analysis of complex signals is indeed a promising and rapidly growing field.

The original results presented hereafter were all obtained in the *FRACTALES* group at Inria, except Theorem 4, which was found in collaboration with Y. Meyer.

2 THEORETICAL TOOLS

In this section we recall some well known notions and present new results on three distinct topics: multifractal analysis (MA), iterated functions systems (IFS) and multifractional Brownian motion (mBm). Due to space limitations and also to the fact that this paper is intended to a multidisciplinary audience, we are more interested in presenting here the most salient results than in going into full details and technical complications. Also, the proves are omitted. Complete descriptions may be found in [38, 14, 46] for each of the topics.

2.1 Multifractal Analysis

As is well known, MA is concerned with the study of the local irregularities of measures or functions, and of their geometrical and statistical distribution.

Useful references are [39, 23, 28, 27], and, in a mathematical framework, [11, 42, 29, 4, 20], where general results have been obtained for deterministic or random *measures*. Other authors extended this type of analysis for studying *point functions* ([6, 33]), which also led to quite complete descriptions. We present here some steps towards a multifractal analysis for *sequences* of *Choquet capacities* with respect to a given *reference measure*.

The motivation for introducing these generalizations are essentially practical :

- In many applications (for instance in signal processing [36]), the relevant quantities for the description of systems cannot be modeled easily by measures (they would not be additive).

Nevertheless, they are usually regular enough to be considered as Choquet capacities. Section 3.1 gives some examples of capacities used in image analysis.

- The advantages of performing a multifractal analysis with respect to any non atomic probability measure μ rather than restricting to the Lebesgue measure are multiple : this generalization for instance "uncovers" some degenerate cases: several measures may be mixed together with the singularities of one of them dominating the others so that we only "see" its spectrum when we use a classical analysis. Also, and this is most important in practical applications, changing the reference measure may lead to much faster convergence rates when estimations are made on real data. We call this type of analysis *mutual multifractal analysis.*

- Finally, the reason for working with sequences is that the notion of resolution is taken into account in a simple manner. Also, most of the results do not need to insure the limit of the sequences. A detailed discussion of the following presentation may be found in [38].

2.1.1 General definitions

We consider here Choquet $\mathcal{B}(E)$-capacities taking values in $[0,1]$, with $E := [0,1[$ and $\mathcal{B}(E)$ being the Borel sets of E. For definitions and properties of Choquet capacities, see for instance [15].

Let $c = (c_n)_{n\geq 1}$ be a sequence of such capacities, and $\mathcal{P} := ((I_j^n)_{0\leq j<\nu_n})_{n\geq 1}$ a sequence of partitions of $[0,1[$ where ν_n is a strictly increasing sequence of intergers. We assume that the following conditions are met :

(C1) $\lim_{n\to\infty} \max_{0\leq j<\nu_n} |I_j^n| = 0$,

(C2) $\forall n, k, \quad I_k^n$ is an interval, semi-open to the right.

We also assume that a non atomic probability measure μ on $[0,1[$ is given. μ will be called the reference measure.

For $x \in [0,1[$ and $n \in \mathbb{N}$, let $I^n(x)$ be the interval I_j^n containing x. Let U_n be the set of indices j such that $c_n(I_j^n)\mu(I_j^n)$ is strictly positive.

2.1.2 Definition of f_h

Let

$$\alpha_n(x) := \frac{\log c_n(I^n(x))}{\log \mu(I^n(x))}$$

which is defined when $c_n(I^n(x))\mu(I^n(x)) \neq 0$, and

$$\alpha(x) := \lim_{n\to\infty} \alpha_n(x)$$

when this limit exists.

We call this quantity the pointwise Hölder exponent of c at point x with respect to μ, although the usual definition involves the limit over all balls centered at x, $c_n = c$ for all n, and $\mu = \mathcal{L}$.

We will use the following definition of dimension of a set E with respect to μ, $\dim_\mu(E)$. This definition is similar to that of Hausdorff dimension [10], except the fact that it is restricted to

coverings by the elements of $\mathcal{P}$.
Let :

$$\begin{aligned}
\mathcal{H}^s_{\mu,\delta}(E) &:= \inf\{\sum_{i=1}^{+\infty} \mu(E_i)^s \;/ E \subset \bigcup_i E_i, \quad |E_i| \le \delta, \quad E_i \in \mathcal{P} \;\; \forall i\} \\
\mathcal{H}^s_{\mu}(E) &:= \lim_{\delta\to 0} \mathcal{H}^s_{\mu,\delta}(E) \\
\dim_\mu(E) &:= \inf\{s \;/\; \mathcal{H}^s_\mu(E) = 0\} = \sup\{s \;/\; \mathcal{H}^s_\mu(E) = +\infty\}
\end{aligned}$$

The f_h multifractal spectrum of c is defined as:

$$f_h(\alpha) := \dim_\mu E_\alpha$$

2.1.3 Definition of $\tilde{f}_g$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$.

$$K^n_\varepsilon(\alpha) := \left\{ k \in \{0, \ldots, \nu_n - 1\} \;/\; \frac{\log c_n(I^n_k)}{\log \mu(I^n_k)} \in [\alpha - \varepsilon, \alpha + \varepsilon] \right\}$$

and

$$N^n_\varepsilon(\alpha) := \mathrm{card}\; K^n_\varepsilon(\alpha)$$

Define, for all $\beta > 0$,

$$\begin{aligned}
S^n_\varepsilon(\alpha,\beta) &:= \sum_{k \in K^n_\varepsilon(\alpha)} \mu(I^n_k)^\beta \\
S_\varepsilon(\alpha,\beta) &:= \limsup_{n\to+\infty} S^n_\varepsilon(\alpha,\beta)
\end{aligned}$$

There exists a real number $\tilde{f}^\varepsilon_g(\alpha)$ such that

$$\begin{aligned}
\beta < \tilde{f}^\varepsilon_g(\alpha) &\implies S_\varepsilon(\alpha,\beta) = +\infty \\
\beta > \tilde{f}^\varepsilon_g(\alpha) &\implies S_\varepsilon(\alpha,\beta) = 0
\end{aligned}$$

$\tilde{f}^\varepsilon_g$ is non decreasing in ε, and $\tilde{f}_g$ is defined by:

$$\tilde{f}_g(\alpha) := \lim_{\varepsilon\to 0} \tilde{f}^\varepsilon_g(\alpha)$$

2.1.4 Definition of f_l

Let $(\lambda_n)_{n\ge 1}$ be a sequence of positive integers such that:

$$\sum_{n>0} \exp(-\eta\lambda_n) < \infty \text{ for all } \eta > 0 \tag{1}$$

Define

$$X_n(x,y) := \sum_{j \in U_n} c_n(I^n_j)^{x+1} \mu(I^n_j)^{-y}$$

and

$$X(x,y) := \limsup_{n\to\infty} \lambda_n^{-1} \log X_n(x,y)$$

Set:

$$\Omega := \{(x,y)/X(x,y) < 0\}$$

A similar argument as one found in [11] allows to show that there exists a concave function ϕ such that :

$$\overset{\circ}{\Omega}= \{(x,y) \in \mathbb{R}^2 \ / \ y < \phi(x-0)\}$$

($\overset{\circ}{\Omega}$ is the interior of Ω).

We suppose that ϕ is finite on an open interval containing 0, and we set :

$$\tau(q) := \phi(q-1)$$

The f_l multifractal spectrum of c is then defined as the function:

$$f_l(\alpha) := \inf_q [q\alpha - \tau(q)]$$

As for the case of measures, the multifractal analysis consists in computing the functions $\alpha_n(x)$ and $\alpha(x)$, and to evaluate and compare the functions f_h, $\tilde{f}_g$ and f_l.

The main results concerning the MA of Choquet capacities are the following ones:

Theorem 1 *Let $c := (c_n)_{n\geq 1}$ be a sequence of Choquet $\mathcal{B}(E)$-capacities defined on $E := [0,1[$, taking values in $[0,1]$, and let $\mathcal{P} := ((I_k^n)_{0\leq k<\nu_n})_{n\geq 1}$ be a sequence of partitions of $[0,1[$ satisfying (C1) and (C2). Then the following inequalities hold:*

$$f_h \leq \tilde{f}_g \leq f_l$$

Note: it is easy to construct examples where $f_h < \tilde{f}_g < f_l$.

Let $\mathcal{C}$ be the space of all Choquet capacities defined on $[0,1[$, and taking values in $[0,1]$.
Let $\mathcal{F}$ be the space of all mappings from $\mathbb{R}^+$ to $[0,1] \bigcup \{-\infty\}$.
Define

$$D(f) := \{\alpha \in \mathbb{R}^+ \ ; f(\alpha) \neq -\infty\}$$

For a closed subset A of $\mathbb{R}^+$, define

$$\begin{aligned}
\mathcal{F}(A) &:= \Big\{f \in \mathcal{F} \ ; D(f) = A \ \text{ and } \ f \text{ is either invertible with } f^{-1} \text{ continuous, or} \\
&\qquad f \text{ is identically zero on } A\} \\
\mathcal{F}^0 &:= \bigcup_{A \text{ closed}} \mathcal{F}(A) \\
\mathcal{F}^1 &:= \{f \in \mathcal{F} \ ; \ \exists \ (f_n)_{n\geq 1}, \ f_n \in \mathcal{F}^0 \text{ for all } n \text{ and } f = \sup_n f_n\}
\end{aligned}$$

Let $\mathcal{S}$ be the set of all functions that are the f_h spectrum of a sequence of capacities belonging to $\mathcal{C}$, i.e.

$$f \in \mathcal{S} \Leftrightarrow \exists \ c := (c_n)_{n\geq 1} \ / \ \forall \ n, \ c_n \in \mathcal{C} \text{ and } f_{h,c} = f$$

The following theorem shows that any element of $\mathcal{F}^1$ is the f_h spectrum of a sequence of $\mathcal{C}$.

Theorem 2

$$\mathcal{F}^1 \subset \mathcal{S}$$

Example 1 *Let F be a F_σ subset of $\mathbb{R}^+$, and $\alpha \in]0,1]$.*
Then

$$\alpha \mathbb{1}_F \in S$$

Theorem 3 *Let $\mathcal{P} := ((I_k^n)_{0\le k<\nu_n})_{n\in\mathbb{N}}$ be a sequence of partitions verifying conditions (C1) and (C2), and $c := (c_n)_n$ a sequence of Choquet capacities.*
Let μ be a reference measure.
Set

$$G_n := \left\{ \frac{\log c_n(I_k^n)}{\log \mu(I_k^n)}\,; 0 \le k < \nu_n \right\}$$

If

$$\lim_n \frac{\log \mathit{card}\ G_n}{\nu_n} = 0$$

then

$$f_{l,c} = \tilde{f}_{g,c}^{\star\star}$$

The condition $\lim_n \frac{\log \text{card}\ G_n}{\nu_n} = 0$ is not necessary. Indeed, it is possible to construct a sequence $c := (c_n)_{n\ge 1}$ of $\mathcal{C}$ which does not verify this condition, and yet is such that $f_{l,c} = \tilde{f}_g^{\star\star} = 1$ for $\alpha \in [0,1[$ and $f_h = 0$ for $\alpha \in [0,1[$.

2.2 Iterated Functions Systems

A field that has been investigated a lot these past years is the theory of Iterated Functions Systems. Although the study of iteration of matrices dates back to Doeblin and Fortet [16] and Dubbins and Freedman [17], it is Hutchinson [30] who really laid the foundations of the IFS theory. Subsequently, several authors have explored this path (see for instance [7, 53, 26, 8] and many others). Barnsley [7] showed that, under some conditions, it is possible to construct an IFS whose attractor is the graph of a continuous nowhere differentiable function. Much more precise results are now known, concerning the almost sure Hölder exponent of such functions [8] or their multifractal spectrum [32, 37].

We are interested here in answering the following question:

Let s be a function from $[0\,;1]$ to $[0,1]$. Under some conditions on s, find practical methods (i.e ones leading to simple implementations) to construct a continuous function f from $[0\,;1]$ to $\mathbb{R}$ such that the Hölder exponent of f at x is exactly $s(x)$ for all x in $[0\,;1]$.

The motivation for this investigation stems partly from applications in signal processing. Indeed, in some cases, it is desirable to model highly irregular signals while precisely controlling the irregularity at each point. This happens, for instance, when the significant information lies more in the singularities of the signal than in its intensity (see section 3.2 for an example in speech modeling). We shall hereafter call α_f the Hölder function of f, which associates, to each point x, the Hölder exponent of the function f at x (for the definition of the Hölder exponent and a detailed description of the following results, see [14]).

Our main result is the following:

Theorem 4 *Let s be a function from $[0,1]$ to $[0,1]$. The following statements are equivalent:*

1. *s is the Hölder function of some continuous functions.*
2. *s is the lower limit of a sequence of continuous functions.*

To go back to IFS, let us consider the following system of functions:

Let $S_i\,(1 \le i \le m)$ be affine transformations represented in matrix notation with respect to (t,x) by:

$$S_i \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ a_i & c_i \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} (i-1)/m \\ b_i \end{pmatrix}$$

We suppose $0 \le t \le 1$ and $1/m < c_i < 1$. Then it is well known that the attractor G of the IFS defined by the S_i's is the graph of a continuous function of (with conditions on a_i and b_i to ensure the continuity of f).

This construction does not allow to control the local regularity at each point, because it can be shown that almost all points have the same Hölder exponent. We thus need to use some sort of generalization of IFS theory, originally due to Anderssson [3]. We consider a collection of sets $(F^k)_{k\in\mathbb{N}^*}$, where each F^k is a non-empty finite set of contractions S_i^k in K, for $i = 0, \ldots, N_k - 1$. We denote by c_i^k the contraction ratio of S_i^k. Then, if some conditions which can be found in [3] are satisfied, the IFS $(K, \{F^k\}_{k\in\mathbb{N}^*})$ possesses a unique attractor. For our purposes, let now F^k be defined as the set of affine transformations $S_i^k\,(0 \le i \le m^k - 1)$, each S_i^k operating only on $[im^{-k+1}\,;(i+1)m^{-k+1}]$ if i is even, and on $[(i-1)m^{-k+1}\,;im^{-k+1}]$ if i is odd. Every S_i maps to $[im^{-k}\,;(i+1)m^{-k}]$. Suppose, also, that we want to interpolate the points $(\frac{i}{m}, y_i)$, for $i = 0, \ldots, m$, $m \ge 2$ and $y_i \in \mathbb{R}$.

If some rather technical conditions are met, we have the following result:

The attractor of the IFS defined above is the graph of a continuous function f such that:

$$f\left(\frac{i}{m}\right) = y_i \quad \forall\, i = 0, \ldots, m$$

and

$$\alpha_f(t) = \liminf_{k \to +\infty} \frac{\log(c^1_{[mt]} \ldots c^k_{[m^k t]})}{\log(m^{-k})} \quad \forall t \in [0\,;1]$$

Where $[x]$ denotes the integer part of x.

In the case of multiple expansions, the one yielding the lower $\alpha_f(t)$ has to be taken.

A further generalization of this construction allows to build an attractor whose Hölder function is any lower limit of a sequence of continuous functions.

We give below two examples, the first one with $\alpha_f(t) = t$ and the second one with $\alpha_f(t) = |\sin(5\pi t)t|$.

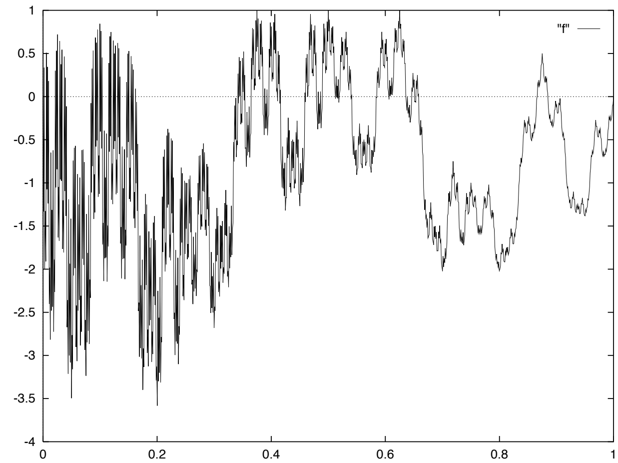


Figure 1: Construction using generalized IFS with $s(t) = t$.

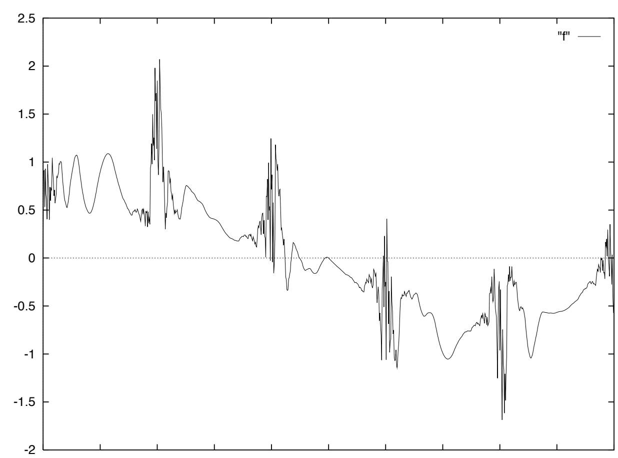


Figure 2: Construction using generalized IFS with $s(t) = |sin(5\pi x)|$.

2.3 Multifractional Brownian Motion

The fractional Brownian motion (fBm) of index H $(0 < H < 1)$ was defined by Mandelbrot and Van Ness (1968) as the stochastic integral, for $t \geq 0$

$$B_H(t) = \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^{0} [(t-s)^{H-1/2} - (-s)^{H-1/2}] dW(s) + \int_0^t (t-s)^{H-1/2} dW(s) \right\},$$

where W denotes a Wiener process defined on $(-\infty, \infty)$.
It is well known that this process has self-similar and stationary increments and continuous sample paths with probability one. Also, with probability one, its graph has Hausdorff and box dimension equal to $2 - H$.

We define hereafter the *multifractional Brownian motion* (mBm) which generalizes the fBm with $t \in [0, \infty[$ by substituting to the parameter H a Hölder function $H(t)$, such that $0 < H(t) < 1$. The mBm provides a useful model for a host of continuous and non stationary natural signals. While the fBm allows to describe processes whose irregularity is the same along all the path, the stationarity property is in practice sometimes restrictive. fBm has for instance been used for synthesizing artificial mountains: the irregularity of such mountains is everywhere the same, which is not realistic because, in particular, erosion's phenomena are not taken in account. It is thus desirable to relax the constraint of stationarity and allow a local control of the irregularity.

Definition 1 (Multifractional Brownian Motion)
Let $H : [0, \infty[\to]0, 1[$ be a Hölder function of exponent $0 < \beta \leq 1$. The following random function is called reduced multifractional Brownian motion with functional parameter H, for $t \in [0, \infty[$:

$$W_{H_t}(t) = \frac{1}{\Gamma(H_t+1/2)} \left\{ \int_{-\infty}^{0} [(t-s)^{H_t-1/2} - (-s)^{H_t-1/2}] dW(s) + \int_0^t (t-s)^{H_t-1/2} dW(s) \right\}.$$

Proposition 1 *$W_{H_t}(t)$, $(0 \leq t < \infty)$ has almost all paths continuous.*

Theorem 5 *Let $(W_H(t))_{t \geq 0}$ denote an fBm of index H $(0 < H < 1)$. Then for any interval $[a, b] \subset]0, 1[$ and $K > 0$ we have almost surely*

$$\lim_{h \to 0} \sup_{a \leq H, H' \leq b \ / \ |H'-H|<h} \ \sup_{t \in [0,K]} |W_H(t) - W_{H'}(t)| = 0.$$

This theorem allows to easily generate sample paths of mBm's.
Assume now that $0 < H(t) < \beta, \forall t \geq 0$.

Proposition 2 *With probability one, for each interval $[a, b] \subset \mathbb{R}^+$, we have:*

$$dim_H \{W_{H_t}, t \in [a, b]\} = dim_B \{W_{H_t}, t \in [a, b]\} = 2 - \min\{H_t, t \in [a, b]\}.$$

where $\dim_H \{W_{H_t}, t \in [a, b]\}$ denotes the Haussdorff dimension of the restriction of the graph of W_{H_t} to $[a, b]$ and $\dim_B \{W_{H_t}, t \in [a, b]\}$ denotes its box dimension.

Proposition 3 *With probability one, the Hölder exponent at point $t_0 \geq 0$ of a multifractional Brownian motion is H_{t_0}.*

We finally show on figures 3 and 4 some sample paths of mBm, along with estimations of the function $H(t)$, using an estimator described in [31].

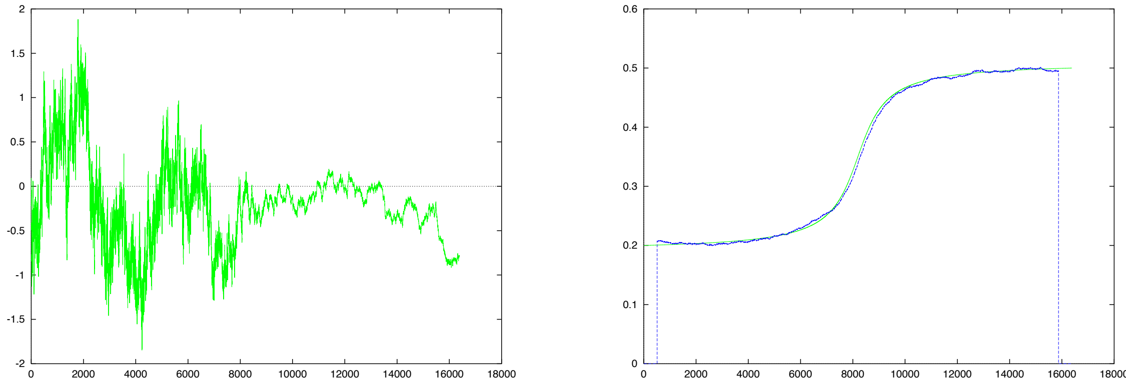


Figure 3: Sample path of mBm : arctan functional parameter

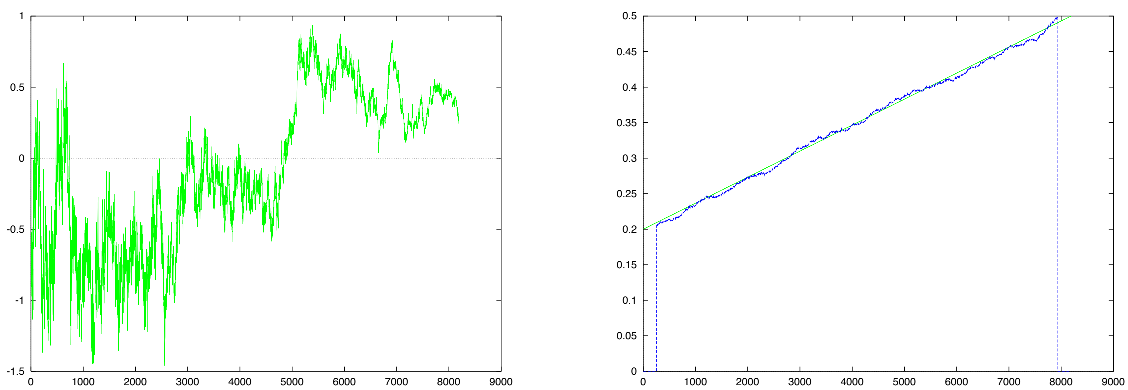


Figure 4: Sample path of mBm : linear functional parameter

3 APPLICATIONS

In this section, we show on three selected cases how the above tools may be applied to partially solve some specific problems in signal processing. Once again, we cannot go here into full technical details; extensive discussions may be found in [36, 35, 37, 13].

3.1 Image Analysis

In the fractal community, "image analysis" usually means computing some fractal dimension from an image representing a certain state of a particular process. The concern here is different, since we want to characterize the image itself in terms of fractal features: in other words, the object of study is the image, and the fractal approach is used to describe its structure.

Image Analysis is an important research field which has a number of applications in robotics, medical imaging, satellite imaging, etc ... We restrict ourselves here to the problem of image segmentation and do not tackle the problem of higher level interpretation. Essentially, image segmentation consists in finding some characteristic entities of an image: these are either described by their contours (edge detection) or by the region where they lie (region extraction).

Edge Detection is by far the most widely used approach. The core of most classical methods is the assumption that edges usually correspond to local extrema of the gradient of the gray levels. In this setting, one has to tackle the problem of computing some "derivative" of a noisy discrete signal. The general idea is to start by smoothing the discrete image data by convolving it with a filter f, and then compute the gradient on the smoothed signal. Edge points are then defined to be the local maxima of the gradient's norm in the gradient's direction. Using additional criteria, one can derive expressions for optimal filters. It is also possible to refine the method using a multiresolution scheme: the original image undergoes a series of successive smoothings, and, at each step, some characteristic points (maxima of the transform) are computed. These points are then used in collaboration through a propagation method, and describe more robustly and accurately the edges.

Though fractal geometry has been introduced a long time ago in image analysis, it is not yet used extensively. Some authors have used the fractal dimension to perform texture classification and image segmentation, others have used correlation or lacunarity. Approaches assuming that a gray level image can be seen as a 3D surface are generally unfounded. This assumption has no theoretical basis, since the scaling properties of the gray levels are generally different from those of the space coordinates. Instead, we should recognize that the gray levels define a measure, which must be studied using a multifractal analysis.

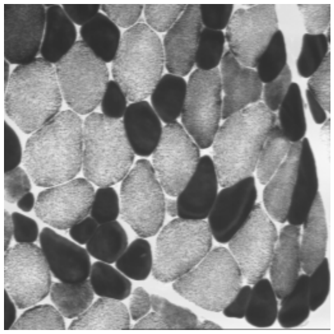
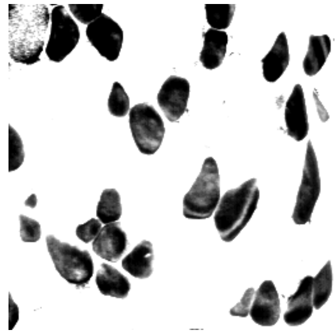
A simple choice is to define the measure μ as the sum of the pixels intensities in the measured region. This will be useful, but not be sufficient for a fine description. One possibility is to use other functions of the gray levels on which we apply the multifractal analysis. Since the notion of resolution is of great importance in image analysis, it is more appropriate to work with set functions than with point functions. However, it occurs that those functions that are relevant here are not in general measures, but capacities. We thus use the multifractal analysis described in section 2.1. Several types of capacities may be used. We only introduce here max, min and iso capacities. If $\Omega^\star$ is the subset of a region Ω where intensity is nowhere zero, $p(i)$ is the intensity at point i, and if $G(\Omega)$ is a reference point in Ω, we define:

$$\begin{aligned}
\mu_{max}(\Omega) &= max_{i\in\Omega}p(i), \\
\mu_{min}(\Omega) &= min_{i\in\Omega^\star}p(i), \\
\mu_{iso}(\Omega) &= \mathrm{Card}\{i \in \Omega / p(i) = p(G(\Omega))\}
\end{aligned}$$

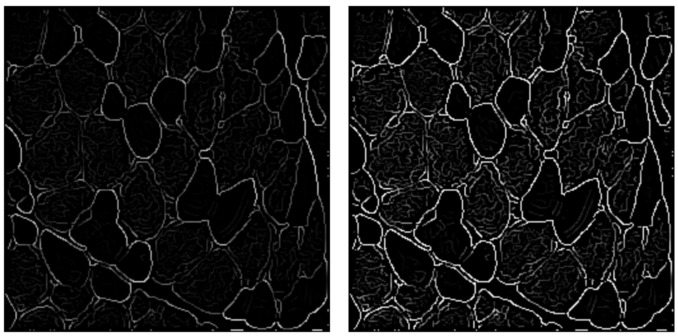
The approach here is, in some sense, inverse to the classical one explained above: instead of smoothing the discrete data in order to compute some derivatives, we stay with the initial discrete values and quantify the singularity at each point. This procedure is based on the idea that, in some cases, it might be impossible to recover an underlying continuous process from the discrete data (if such a process exists ...). Thus it seems more natural to directly model the sampled signal. The advantage is that no information is lost or introduced by smoothing. The drawback is that this method may well be much more sensitive to noise. This is why several capacities have to be used in cooperation. Using jointly the local information provided by α and the global one contained in $f(\alpha)$, it is possible to construct an operator on the image which is idempotent and reacts differently to different types of singularities (provided that the noise is not too important).

A characteristic feature of edges is that they usually correspond to "rare" events. In other words, if too many edges are detected in a portion of an image, then the human visual system will have a tendency to see a textured zone, rather than of a concentration of edges. Remember that $f_g(\alpha)$ measures, loosely speaking, how rare or frequent an event of singularity α is. Assuming that $f_g(\alpha)$ and $f_h(\alpha)$ are equal, we may assess how "rare" a smooth edge is, because a smooth edge point will belong to a set E_α whose dimension is one: this is a simple use of the connection between geometry and probability provided by the assumed equality between f_g and f_h: from a geometrical point of view, a point with prescribed singularity belongs to a set of given $f_h(\alpha)$. If $f_h = f_g$, then $f_g(\alpha)$ is also given, and the probability of finding such a point in the image at a fixed resolution is known. An edge is thus characterized both by a local condition (α), and a global one ($f(\alpha)$).

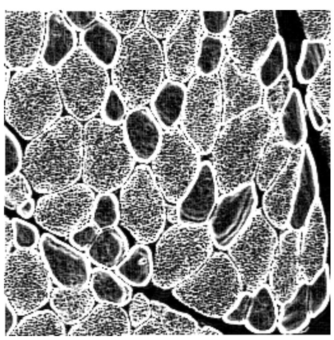
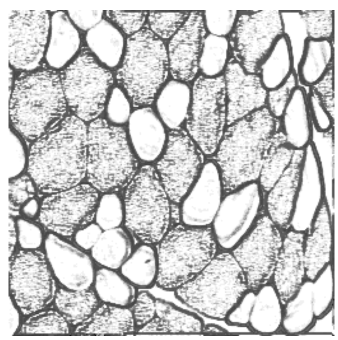
Results obtained with this approach are presented on a medical image and a natural scene below.

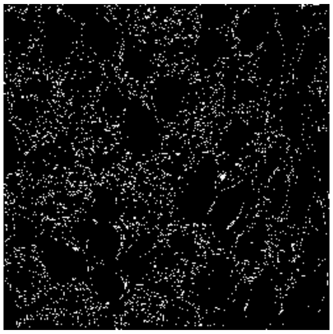
Muscle image Brightened image



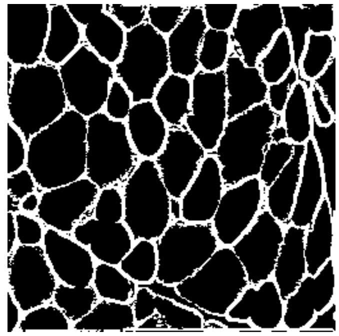
Two results of segmentation with a gradient method

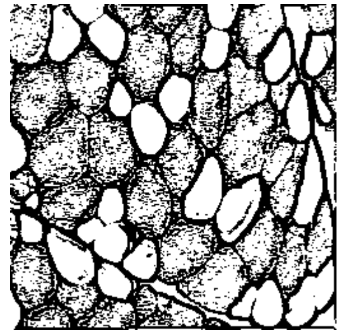
Hölder exponent with max capacity Related dimension image



Irregular points ($f(\alpha) \ll 1$)



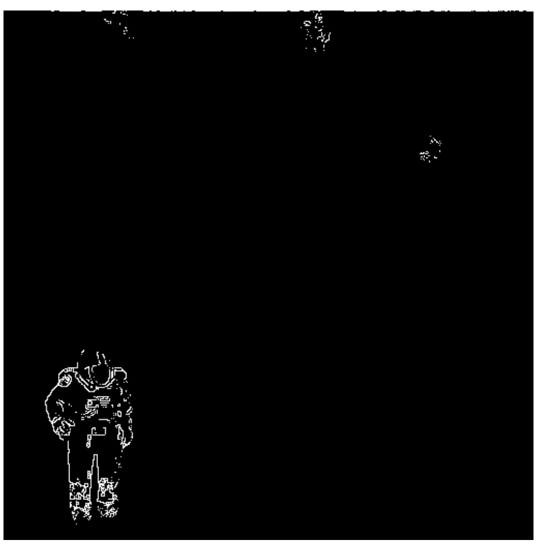
Contour points ($f(\alpha) \simeq 1$)



Regular points ($f(\alpha) \simeq 2$)

Original image

Enhanced image



Points on smooth edges

Points on irregular edges

These results demonstrate that the use of a multifractal characterization of image points can help to solve the problem of segmentation. In several cases, this approach gives at least as good results as the classical ones.

3.2 Speech Synthesis

We are interested here in the general problem of speech signal synthesis. The precise setting is the following: the French CNET (Centre National d'Etudes en Télécommunication) has developed a synthesis algorithm, PSOLA [9], based on the concatenation of acoustic units stored in a dictionary. The latter are obtained by segmentation of longer acoustic units, the logatomes, previously recorded by a human operator. Although this method gives very good results, there subsists the problem of the dictionary construction: three months are necessary for the recording and the segmentation of the 1200 logatomes needed in French. Each time a new voice is being created, one has to go through a complex and time consuming process.

This problem could be solved by using existing dictionaries to create new "voices". A simple idea is to perform interpolations between corresponding logatomes of two dictionaries coming from two different voices in such way that at each step of the interpolation, the signal remains a logatome. A first step towards this goal is to obtain a robust functional representation of each logatome. This is the particular problem we address hereafter.

Speech sounds are in some cases produced by turbulence phenomena [12]. It is well known that several aspects of turbulence are amenable to a multifractal analysis. This and other theoretical considerations, as well as the general aspect of the signals (see fig. 5), motivate the use of a multifractal approach for speech signal modeling.

There are two main ideas in the fractal model: the first one is that there exists in a speech signal a set of remarkable points, where the acoustic information is specially relevant (for instance the points that define the waveform). These points we call tag points. The second idea is that the singularities (Hölder exponents) at each point of the signal play a key role in the determination of the voice texture. A simple and efficient tool for controlling both tag points and singularities is fractal interpolation using IFS, as defined in section 2.2.

The method involves two steps: determination of the interpolation points, which will control the general shape of the attractor, and computation of the interpolation functions, which will take care of the local singularity. Different procedures are used for voiced and unvoiced parts of the signal: voiced sections are quasi-periodic signals, as unvoiced parts are much more irregular.

A result on the sound /Seu/ is presented below. Auditive comparison of the two signals shows that the fractal interpolation is a very faithful reproduction of the original sound.

3.3 Road Traffic Modeling

In the last decades, the growth of road traffic has become more and more important, yielding enormous time and energy wasting, noise, pollution and accidents, and giving rise to health and financial problem. The necessity of a good understanding of the traffic is thus essential, at all scales.

A classical and simple modeling of the traffic flow leads to a Burger's equation involving the density $\rho(x,t)$ of vehicles at time t at position x.

When the initial density $\rho_0(x)$ is smooth, it is well known that, in the inviscid limit, the solution $\rho(x,t)$ is smooth except for isolated discontinuities, called *shocks*. For long times, the smooth zones become ramps of slope $1/t$. When $\rho_0(x)$ is a fractional Brownian motion, the behavior of the solutions is very different ([51, 50]). The self-similarity of the initial conditions has some drastic consequences. Intuitively, for smooth initial conditions, it takes some times for shocks to be created;

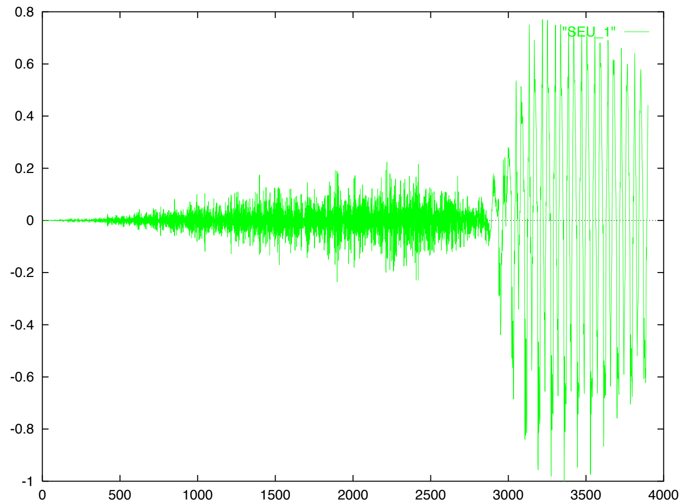


Figure 5: Original sound /seu/.

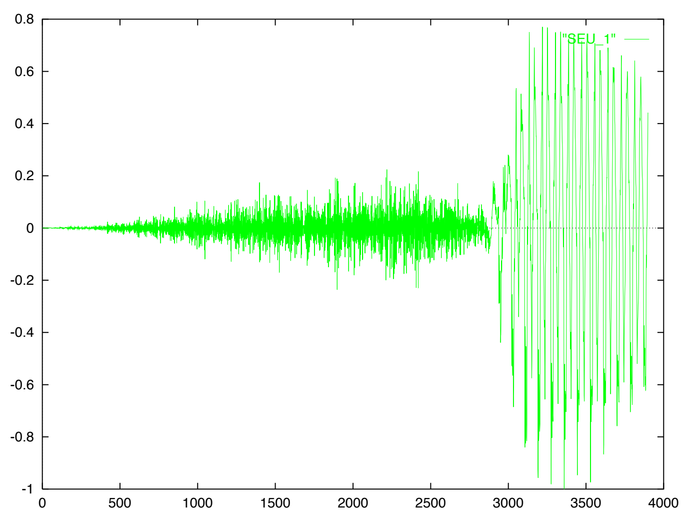


Figure 6: Synthesis of the sound /seu/ with generalized IFS.

as for scaling initial conditions, shocks may appear after arbitrarily short times .

Figure 7 displays data representing the traffic flow over one day at Porte de Bercy, on the Boulevard Périphérique of Paris. Figure 8 shows a zoom on a few hours.

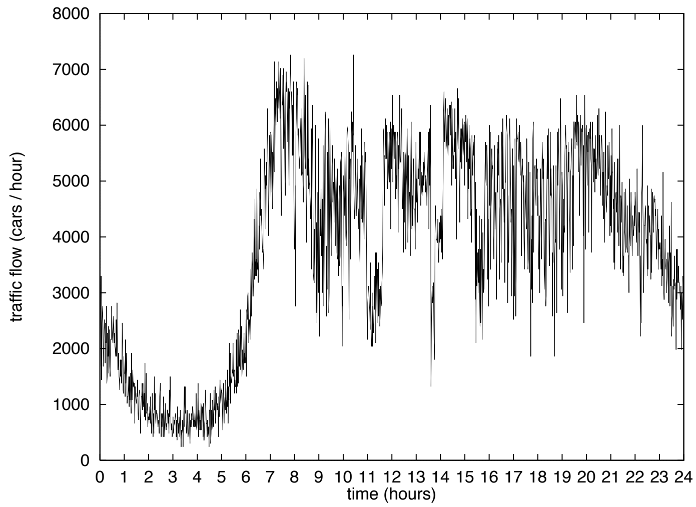


Figure 7: Traffic flow on the Boulevard Périphérique of Paris.

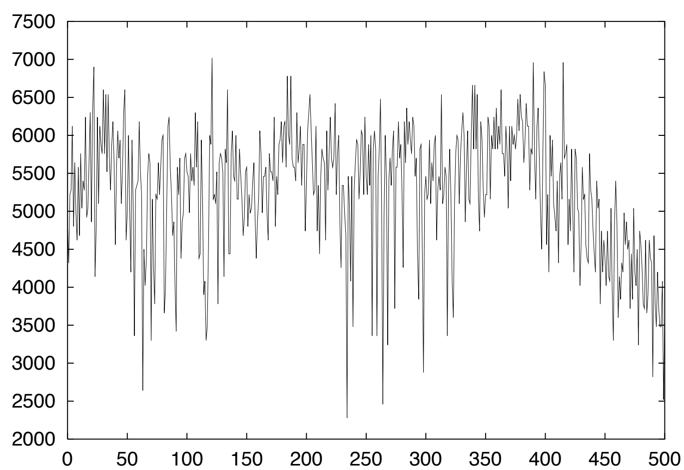


Figure 8: Zoom on the traffic flow.

These graphs suggest that the real situation might be even “worse” than the one obtained by solving Burger’s equation with scaling initial conditions. The traffic signal, when properly normalized, bears some ressemblance with an mBm. In particular, the mBm model allows to take into account the non sationnarity of the data. Furthermore, preliminary investigations indicate that it might also be used to perform short term prediction.

4 Conclusion

Much some work is obviously needed in the field of fractal analysis of signals. However, the results presented in section 3 show that this approach allows to obtain new and interesting descriptions of complex signals. In some situations, it has even already yielded better results than "classical" methods.

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